Linear Algebra I
Exercise Sheet 1: Vector Spaces

For each of the following determine whether V is a vector space. Standard addition and scaling is to be assumed unless otherwise stated. If V is a vector space then determine a basis.

Question 1. $V = \phi$, the empty set.

Question 2. $V = \mathbb{R}$.

Question 3. $V = \mathbb{R}^n$, $n \geq 2$.

Question 4. $V = \mathbb{C}$.

Question 5. Let k be prime.

$$V = \mathbb{Z}_k = \big\{\{nk \,|\, n \in \mathbb{Z}\}, \{nk+1 \,|\, n \in \mathbb{Z}\}, \dots, \{nk+k-1 \,|\, n \in \mathbb{Z}\}\big\},$$

where

$$\{n_1k+p\} + \{n_2k+q\} = \{(n_1+n_2)k+p+q\}$$

for all $p, q \in \mathbb{Z}_k$ and

$$\lambda \cdot \{nk+p\} = \{\lambda nk + \lambda p\}$$

for all $\lambda \in \mathbb{Z}_k$.

Question 6. Let k be composite (not prime).

$$V = \mathbb{Z}_k = \big\{\{nk \,|\, n \in \mathbb{Z}\}, \{nk+1 \,|\, n \in \mathbb{Z}\}, \dots, \{nk+k-1 \,|\, n \in \mathbb{Z}\}\big\},$$

where

$$\{n_1k+p\} + \{n_2k+q\} = \{(n_1+n_2)k+p+q\}$$

for all $p, q \in \mathbb{Z}_k$ and

$$\lambda \cdot \{nk+p\} = \{\lambda nk + \lambda p\}$$

for all $\lambda \in \mathbb{Z}_k$.

Question 7. $V = Z_k^n$ for $n > 1$ separately for k prime and k composite.

Question 8. $V = \{a_0 + a_1x + \ldots + a_ix^i \,|\, i \leq n\}$, for some fixed $n \in \mathbb{N}$, where $a_i \in \mathbb{R}$ for all i. Addition and scaling are standard for polynomials with real coefficients.

Question 9. $V = \{a_0 + a_1x + \ldots + a_nx^n \,|\, n \in \mathbb{N}\}$.

Question 10. V is the space of differentiable functions in one variable $f : \mathbb{R} \to \mathbb{R}$.

Question 11. V is the space of integrable functions in two variables from $g : \mathbb{C}^2 \to \mathbb{C}^2$

Question 12. $V = \big\{a(n) \,|\, a(n+2) = a(n+1) + a(n),\ n \geq 0,\ a(0), a(1) \in \mathbb{R}\big\}$.

Question 13. V is the space of $m \times n$ matrices with complex entries. Scaling occurs by multiplying each entry of the matrix by the scalar.

Question 14. $V = \{L \,|\, L : W_1 \to W_2 \text{ is a linear map}\}$.